

A Realization of the infinite-dimensional Symmetries of Conformal Mechanics

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Abstract

We discuss the possibility of realizing the infinite dimensional symmetries of conformal mechanics as time reparametrizations, generalizing the realization of the $SL(2, \mathbf{R})$ symmetry of the de Alfaro, Fubini, Furlan model in terms of quasi-primary fields. We find that this is possible using an appropriate generalization of the transformation law for the quasi-primary fields.

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I. INTRODUCTION

Recently, much attention has been devoted to the study of the symmetries of conformal mechanics [1–3], especially because this model is expected to play an important role in the context of the $\text{AdS}_2/\text{CFT}_1$ correspondence [3,4]. Another reason of the renewed interest in this model is the fact that it describes the motion of a particle near the horizon of the Reissner-Nordstrom black hole [5].

Starting from the symplectic symmetry of its phase space, it was shown that conformal mechanics admits an infinite set of conserved charges, which span a w_∞ algebra [1,2]. However, the physical interpretation of this algebra and of the related charges, and its connection with the conformal algebra of the original de Alfaro, Fubini, Furlan (DFF) model [6] has not been discussed in the literature. In particular, the Virasoro algebra appearing as subgroup of the w_∞ algebra can be thought of as an infinite dimensional extension of the $SL(2, \mathbf{R})$ symmetry of the DFF model. It seems, therefore, very natural that a realization of the infinite dimensional symmetry of conformal mechanics exists, which generalizes the realization of the $SL(2, \mathbf{R})$ symmetry of the DFF model in terms of quasi-primary fields. The $SL(2, \mathbf{R})$ symmetry of the DFF model is realized as conformal transformations of the time t with the coordinate q transforming as a quasi-primary conformal field [6]. The most natural generalization of this realization leading to an infinite dimensional symmetry is obtained by requiring invariance under diffeomorphisms of the time t . Such a generalization is also consistent with the fact that the conformal group in one dimension can be realized as the group of time reparametrizations and is generated by a Virasoro algebra [3]. An implementation of the previous ideas could help to shed some light on the mysteries of the $\text{AdS}_2/\text{CFT}_1$ correspondence.

In this paper we discuss in detail these points. In particular, we show how the realization of the $SL(2, \mathbf{R})$ symmetry of the DFF model can be generalized to an infinite dimensional symmetry realized as time reparametrizations. The price we pay in doing this operation is that the transformation law for the coordinate q is not given by the simple transformation

law of a conformal field with given weight but contains further terms depending on the dynamical variables of the system.

Another, related, issue we will address in this paper is to understand how a infinite dimensional symmetry can arise in a model with only one degree of freedom. We will show that the w_∞ and conformal algebras have different origin, and that the generators of w_∞ are not functionally independent, but are functions of two basic conserved quantities.

The structure of the paper is the following. In sect. 2 we discuss the conformal symmetry of conformal mechanics and its realization in terms of time reparametrizations generalizing the DFF realization of the $SL(2, \mathbf{R})$ symmetry. In sect. 3, we show how this symmetry can be further extended to the full w_∞ algebra.

II. SYMMETRIES OF ONE-DIMENSIONAL CONFORMAL MECHANICS

In Ref. [1] was observed that the algebra of the symmetries of a generic one-dimensional conformal model described by the Hamiltonian,

$$H = \frac{p^2}{2f(u)}, \quad (1)$$

where $f(u)$ is a function of $u = pq$, can be extended to a Virasoro algebra with generators

$$L_n = -\frac{1}{2}q^{1+n}p^{1-n}f^n,$$

such that¹

$$\{L_n, L_m\} = (m - n)L_{m+n}.$$

The generators L_n are not conserved charges. The origin of the symmetry is rather obscure in this formulation, but was partially clarified in Ref. [2], where the symmetry was also extended to a w_∞ algebra. Since the Hamiltonian (1) can be transformed into the free

¹With $\{F, G\}$ we mean the Poisson Brackets with the convention $\{q, p\} = 1$.

Hamiltonian $H = p^2/2$ by a simple canonical transformation, the symmetries of the action are easily found. It turns out that the diffeomorphisms which preserve the symplectic form $\Omega = dp \wedge dq - dH \wedge dt$ are generated by vector fields spanning the w_∞ algebra of area-preserving diffeomorphisms [7]. One can choose a basis of generators

$$v_m^l = \left(\frac{p}{\sqrt{f}} \right)^{l+1} \left(q\sqrt{f} - \frac{p}{\sqrt{f}}t \right)^{l+m+1}, \quad (2)$$

with $l = 0, 1, 2, \dots$ and $m \in \mathbf{Z}$, which satisfy the w_∞ algebra:

$$\{v_m^l, v_{m'}^{l'}\} = [m(l' + 1) - m'(l + 1)]v_{m+m'}^{l+l'}. \quad (3)$$

The v_m^l are conserved charges (constants of motion):

$$\frac{dv_m^l}{dt} = \frac{\partial v_m^l}{\partial t} + \{v_m^l, H\} = 0. \quad (4)$$

The physical interpretation of these charges is still not completely evident. However, the same result can be recovered in a more intuitive way, when one tries to extend to an infinite dimensional Virasoro algebra the $SL(2, \mathbf{R})$ symmetry algebra of the conformal mechanics of DFF [6], whose Hamiltonian

$$H_{DFF} = \frac{p^2}{2} + \frac{g}{2q^2}, \quad (5)$$

with g constant, is a special case of the model (1) with $f = (1 + g/u^2)^{-1}$. The coordinate q transforms under the conformal group $SL(2, \mathbf{R})$ as a quasi-primary field:

$$q'(t') = \frac{1}{\gamma t' + \delta} q(t), \quad t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \alpha\delta - \beta\gamma = 1. \quad (6)$$

The generators of infinitesimal translations, dilatations and special conformal transformations are given by H_{DFF} , D_{DFF} and K_{DFF} , where

$$\begin{aligned} D_{DFF} &= tH - \frac{pq}{2}, \\ K_{DFF} &= t^2H - tpq + \frac{q^2}{2}. \end{aligned} \quad (7)$$

These generators are conserved under time evolution. The conformal $SL(2, \mathbf{R})$ algebra is

$$\{H_{DFF}, D_{DFF}\} = H_{DFF}, \quad \{K_{DFF}, D_{DFF}\} = -K_{DFF}, \quad \{H_{DFF}, K_{DFF}\} = 2D_{DFF}. \quad (8)$$

One can show that it is impossible to find an infinite dimensional extension of the little algebra generated by $H_{DFF} = L_{-1}$, $D_{DFF} = L_0$, $K_{DFF} = L_1$, which is realized as time reparametrizations and preserves the transformation law for the coordinates q given in Eq. (6). In fact, since $\{L_{-1}, L_0, L_1\}$ are conserved charges, any extension can consist only of conserved charges. This proposition is easily proven. From

$$\begin{aligned} \{L_n, L_{-1}\} &= -(n+1)L_{n-1} \\ \{L_n, L_1\} &= -(n-1)L_{n+1} \\ \frac{dL_{-1}}{dt} &= \frac{dL_0}{dt} = \frac{dL_1}{dt} = 0, \end{aligned} \quad (9)$$

we conclude that

$$\frac{dL_{-2}}{dt} = \frac{dL_2}{dt} = 0 \quad \Rightarrow \quad \frac{dL_n}{dt} = 0, \quad n \leq -2 \quad \text{and} \quad n \geq +2. \quad (10)$$

The first relation on the r.h.s. of the previous implication is proved considering the total time derivative of

$$\{L_2, L_{-1}\} = -3L_1$$

and the last line of (9). From $\frac{d}{dt}L_n = 0$ for $n \geq -1$, we infer that also the remaining generators L_n , $n \leq -2$ are conserved. But this conclusion implies that the coordinate q should transform as a primary (not only quasi-primary) field in order to give a conserved action modulo a total time derivative, which is not the case. In fact, if we consider a generic time reparametrization $t' = h(t)$ we have

$$q'(t') = \left(\frac{dh(t)}{dt} \right)^{1/2} q(t). \quad (11)$$

The variation of the action $S = \int dt L(q, \dot{q})$ is given by

$$\delta S = \frac{1}{2} \int dt \left[\frac{d}{dt} \left(\frac{\ddot{h}}{\dot{h}} \frac{q}{2} \right) - (h, t) \frac{q^2}{2} \right], \quad (12)$$

where (h, t) is the Schwarzian derivative of the map $t \rightarrow t' = h(t)$. It is easy to see that δS is a total derivative only if $h(t)$ is a conformal map, while for a generic infinitesimal time transformation $t' = t + \epsilon g(t)$, Eq. (12) becomes:

$$\delta S = \frac{1}{2} \int dt \epsilon \ddot{g} q \dot{q} . \quad (13)$$

The conclusion is that there cannot exist conserved charges corresponding to time-reparametrizations with the field q transforming as in Eq. (11). This implies, in turn, that it is impossible to generalize the $SL(2, \mathbf{R})$ conformal symmetry (11) to the full diffeomorphisms group, if one requires that q transforms as a conformal field.

It is evident that the quasi-primarity condition (6) is too strong. Let us try to relax this condition and impose an analogous condition only for translations and dilatations. From Noether theorem we get $D = tH - pq/2$ ($= D_{DFF}$), which in turn implies

$$\frac{dD}{dt} = 0 \Rightarrow H = \frac{p^2}{2f(pq)} . \quad (14)$$

Hence, H must have the form given by Eq. (1) with $f(pq)$ being an arbitrary function. If we further impose that q transforms as in (6) (quasi-primarity) for infinitesimal special conformal transformation of t , we are forced to take

$$\begin{aligned} H &= \frac{p^2}{2} \left(1 + \frac{g_1}{pq} + \frac{g_2}{(pq)^2} \right) \\ K &= t^2 H - tpq + \frac{q^2}{2} - \frac{g_1}{2} t, \end{aligned} \quad (15)$$

where g_1 and g_2 are constants. With the trivial canonical transformation $p' = p + g_1/(2q)$, $q' = q$, we have $H \rightarrow H_{DFF}$ with $g = g_2 - g_1^2/4$, so the quasi-primarity condition for the q is fulfilled only by the DFF conformal mechanics. Let us assume instead that under infinitesimal special conformal transformations $t' = t - \omega t^2$, q changes according to

$$\delta_0 q = \omega \{q, K\} = \omega t^2 \dot{q} - \omega t q + \text{corrections} \quad (16)$$

where $\delta_0 q = q'(t) - q(t)$ and K is the (conserved) charge to be determined. If we take $K = t^2 H - tpq + f_2$, the condition $\frac{d}{dt} K = 0$ translates into

$$\frac{d}{dt}f_2 = pq \Rightarrow \begin{cases} f_2 = \frac{q^2}{2} & \text{DFF solution} \\ f_2 = \frac{(pq)^2}{4H} \end{cases} \quad (17)$$

If we choose the second solution, we can write the three conserved generators as:

$$\begin{aligned} L_{-1} &= H_{CCM} = H \left(t - \frac{pq}{2H} \right)^0, \\ L_0 &= D_{CCM} = H \left(t - \frac{pq}{2H} \right)^1, \\ L_1 &= K_{CCM} = H \left(t - \frac{pq}{2H} \right)^2. \end{aligned} \quad (18)$$

The extension to a Virasoro Algebra of conserved charges is now immediate:

$$\begin{aligned} L_n &= H \left(t - \frac{pq}{2H} \right)^{1+n} \quad n \in \mathbf{Z} \\ \frac{d}{dt}L_n &= 0 \\ \{L_n, L_m\} &= (m - n)L_{n+m}. \end{aligned} \quad (19)$$

The meaning of these charges is clear from their very construction: they canonically implement the transformation of the variables q, p as primary “fields” plus interaction-dependent corrections necessary to close a Virasoro algebra of conserved quantities. It can be noted that the DFF generators of the $SL(2, \mathbf{R})$ algebra coincide with the generators (18) only in the non-interacting $g = 0$ case.

According to this interpretation we can establish a correspondence between the charges L_{n-1} and the infinitesimal time transformations $t \rightarrow t - \omega t^n$, with $n \in \mathbf{Z}$, identifying the L_{n-1} with the Noether charges for such transformations. In fact, under L_{n-1} the coordinates and momenta transform as

$$\begin{aligned} \delta_0 q &= \omega \{q, L_{n-1}\} \\ \delta_0 p &= \omega \{p, L_{n-1}\}. \end{aligned} \quad (20)$$

Using Eq. (20) one can easily find the transformation law of p, q for a generic infinitesimal time transformation $t' = t - \epsilon(t)$,

$$\begin{aligned}\delta_0 q &= \left\{ q, H\epsilon \left(t - \frac{pq}{2H} \right) \right\} \\ \delta_0 p &= \left\{ p, H\epsilon \left(t - \frac{pq}{2H} \right) \right\}.\end{aligned}\tag{21}$$

Expanding $\epsilon(t) = \sum_n \epsilon_n t^n$, and computing the Poisson brackets, we get

$$\begin{aligned}\delta_0 q &= \left(\epsilon \dot{q} - \frac{1}{2} \dot{\epsilon} q \right) + \sum_n \epsilon_n \sum_{k \geq 2} \binom{n}{k} t^{n-k} \left\{ q, H \left(-\frac{pq}{2H} \right)^k \right\}, \\ \delta_0 p &= \left(\epsilon \dot{p} + \frac{1}{2} \dot{\epsilon} p \right) + \sum_n \epsilon_n \sum_{k \geq 2} \binom{n}{k} t^{n-k} \left\{ p, H \left(-\frac{pq}{2H} \right)^k \right\}.\end{aligned}\tag{22}$$

Comparing this realization of the infinite dimensional conformal symmetry with the usual realization in terms of primary fields,

$$\delta_0 q = \epsilon \dot{q} - \frac{1}{2} \dot{\epsilon} q, \tag{23}$$

one easily sees that our realization of the infinite dimensional conformal algebra differs from the usual one for terms depending on the dynamics of the system. In particular, in contrast with ordinary "spacetime" symmetries, the additional terms depend both on q and \dot{q} .

III. FURTHER EXTENSIONS OF THE SYMMETRY ALGEBRA

The Virasoro algebra (19) can be easily extended to a w_∞ -algebra analogous to (3). In fact, starting from the two elementary conserved charges H and $G = t - \frac{pq}{2H}$, one can define the conserved generators

$$w_{l,m} = H^{1+l} G^m \quad l, m \in \mathbf{Z},$$

with Poisson brackets

$$\{w_{l,m}, w_{l',m'}\} = [m'(l+1) - m(l'+1)] w_{l+l', m+m'}.\tag{24}$$

The charges so defined satisfy the same algebra as (3), and essentially coincide with the charges v_m^l , modulo some redefinitions. The Virasoro algebra (19) is contained in (24) as a subalgebra, $L_n = w_{0,n+1}$.

It should be noticed that in the particular case of DFF conformal mechanics, the algebra (24) also includes the generator of special conformal transformations K_{DFF} . In fact, from the definitions (7) and (19),

$$K_{DFF} = K_{CCM} + \frac{g}{4H_{DFF}} = w_{0,2} + \frac{g}{4}w_{-2,0}. \quad (25)$$

Our construction clarifies how an infinite set of conserved charges can originate from a system with only one degree of freedom: all the charges are in fact functionally dependent from the two elementary charges H and G .

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